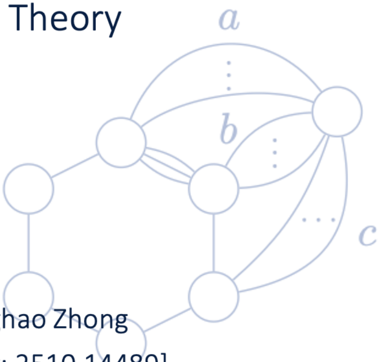
Algebraic of Supersymmetric Quantum Field Theory



Lucy (Leyi) Jiang

ASB Group Meeting - 13 Nov 2025

Based on work with Jazz E. Z. Ooi, Richard Stone and Zhenghao Zhong

Bootstrapping Mirror Pairs: The Beginning of the End [arXiv: 2510.14489]

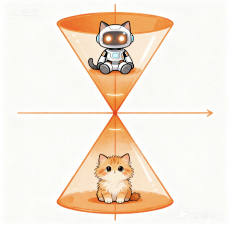
UNIVERSITY OF OXFORD Mathematical Institute

CENTER FOR SYSTEMS BIOLOGY DRESDEN

A (Little) Physics Background

In the 20th century, we saw huge shifts in Physics:

Quantum Mechanics
(Physics when things
are small)



Special Relativity
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[Image generated by AI]

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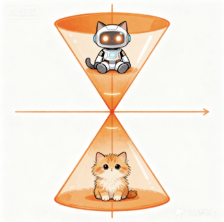

Special Relativity
[Physics when things are fast]

Quantum Field Theory (QFT)
[Particles are field excitations]

[Image generated by AI]

Supersymmetric QFT

- Supersymmetry (SUSY): A symmetry that relates bosons (force carriers) and fermions (matter particles)
- In particular, we work in $3d$ $\mathcal{N} = 4$ supersymmetry.
  - There are 2 space dim $+$ 1 time dim.
  - This is a theory with 8 supercharges. (The more supercharges, the more supersymmetry)

Moduli Space of Vacua

We are interested in the background spacetime from which particle excitations happen.

- **Moduli space of vacua**: a geometric space that parametrises all possible ground states of a theory.
- Supersymmetric vacua satisfies *the Vacuum Condition* of scalar potential:

$$V(\phi_V, \phi_H) = 0$$

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Mathematically, $V(\phi_V, \phi_H) = 0$ is typically a system of polynomial equations. Hence, the moduli space of vacua can often be understood as an algebraic variety.

Moduli Space of Vacua

There are two key branches of the moduli space of vacua:

- **Coulomb Branch** ($\mathcal{M}_{\mathcal{C}}$): vector multiplet scalars acquire expectation values,

$$V(\phi_V \neq 0, \phi_H = 0) = 0.$$

- **Higgs Branch** ($\mathcal{M}_{\mathcal{H}}$): hypermultiplet scalars acquire expectation values,

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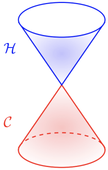
$\mathcal{M}_{\mathcal{C}}$ and $\mathcal{M}_{\mathcal{H}}$ are both hyperkähler manifolds. They intersect at a single point, $V(\phi_V = 0, \phi_H = 0) = 0$, the super-conformal IR fixed point.



Duality

Intriligator and Seiberg (1996) $3d$ $\mathcal{N} = 4$ theories X, Y are $3d$ mirror symmetry to each other if:

$$\mathcal{M}_{\mathcal{C}}^{X} = \mathcal{M}_{\mathcal{H}}^{Y}$$
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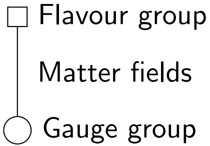
Challenge: How do we systematically find mirror symmetry pairs?

Quiver Gauge Theory

Quiver gauge theory are supersymmetric QFT theories that can be combinatorially represented with **quiver diagrams**.
For $3d$ $\mathcal{N} = 4$ theories, the quiver diagram is an undirected graph:

- Circular nodes represent the gauge groups of the theory;
- Square nodes represent the flavour symmetry groups;
- Edges connecting these nodes represent matter fields.



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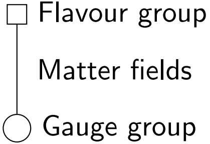
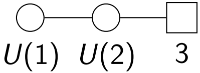


For example,



Finding 3d mirror pairs

Challenge: How to we systematically find mirror symmetry pairs?
Brane constructions: quiver gauge theories can be built using physical objects called branes in string theory.

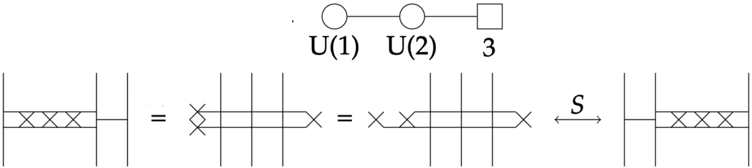


Figure: An example of $3d$ mirror symmetry construction via brane setups.

Limitation: the set up is hard to generalise beyond linear quivers.

Finding 3d mirror pairs

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Brane constructions: quiver gauge theories can be built using physical objects called branes in string theory.

New approach: A combinatorial bootstrapping method. A quartet of algorithms that "move" you through the space of quiver gauge theories.

	Higgs branch	Coulomb branch
Higgsing	**Quiver Subtraction**	**Decay and Fission**
UnHiggsing	**Quiver Addition**	**Growth and Fusion**

Figure: Quiver algorithms

Quiver Algorithms

Hasse diagram: a graph that represents a partially ordered set. In our context, it's the "family tree" of related quantum field theories.
The partial ordering: Theory A is "greater than" Theory B if Theory B can be obtained from Theory A via a Higgs mechanism.

- Moving down the diagram = Higgsing (breaking symmetry).
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Depending on which scalar fields acquire the expectation values, we can distinguish Coulomb branch Higgsing and Higgs branch Higgsing.

	Higgs branch	Coulomb branch
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Quiver Algorithms

Starting with a known $3d$ mirror pair (X, Y). Performing the quiver algorithms in pairs preserves $3d$ mirror symmetry.

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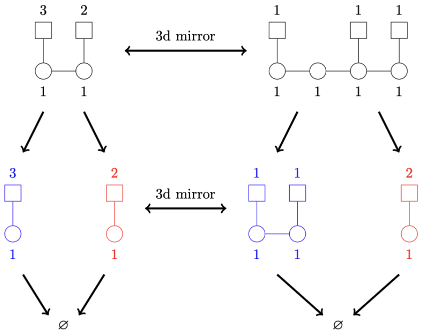
For example, we perform Decay and Fission to the left quiver and Quiver Subtraction to the right. The corresponding (same level, same colour) quivers are $3d$ mirror pairs.



Quiver Algorithms

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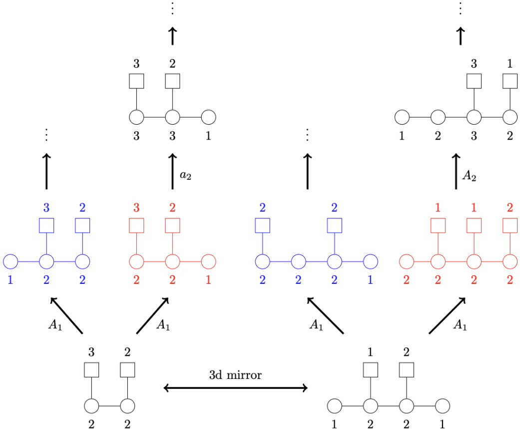
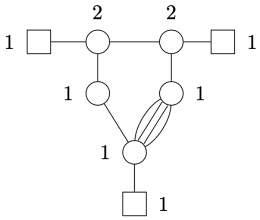


Bootstrapping 3d Mirrors



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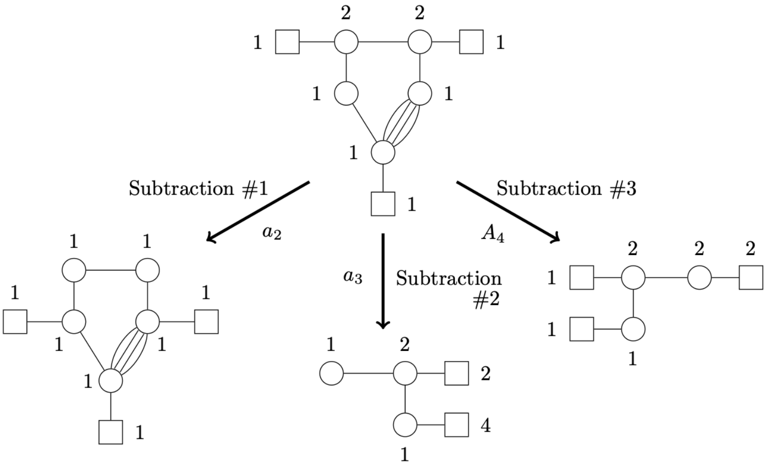

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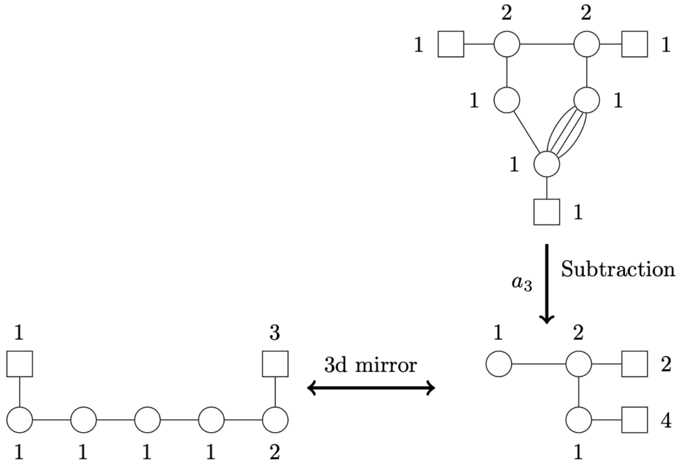

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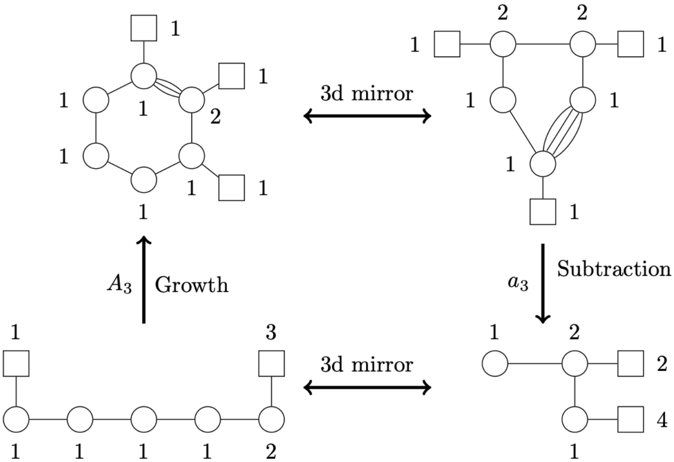
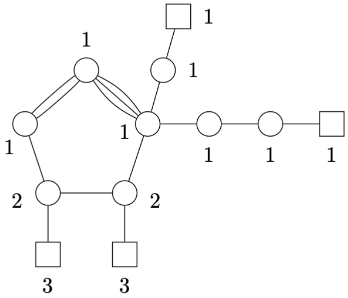


A New Playground: Sunshine Quivers

Sunshine quivers are characterised by two distinct components:

- A Central Loop: A cycle of gauge nodes.
- Rays: Linear chains of nodes emanating from the loop.



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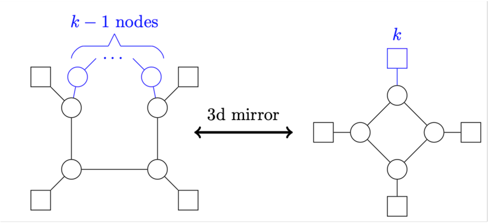
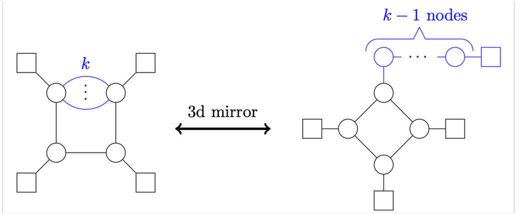
The distance between rays is dual to the number of flavours in the mirror quiver.





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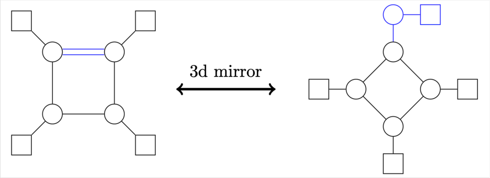

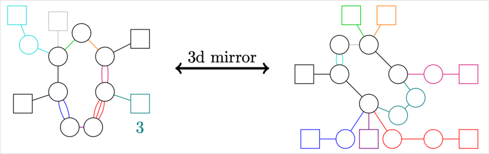

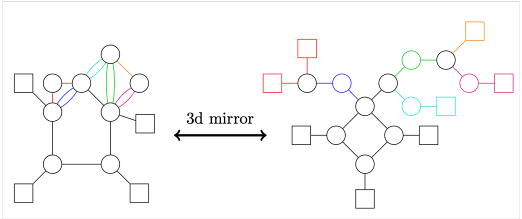

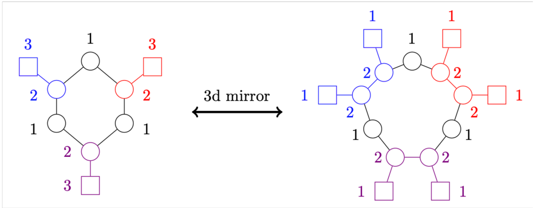

Conclusion and Outlook

Summary:

- We introduced a combinatorial, brane-free framework for 3D mirror symmetry.
- This bootstrap method can find mirrors for given quivers and generate new pairs.
- A rich new class of circular sunshine quivers.

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Future Directions:

- Computer Implementation: Code the algorithms to systematically implement quiver algorithms.
- Generalise Algorithms: Extend the quartet to include SU, SO and Sp gauge groups.
- Beyong $3d$: What is the implications of our methods and results for understanding other theories with 8 supercharges?

The universe is an enormous direct product of representations of symmetry groups.

— Steven Weinberg

Thank you!

Any questions?

Additional Slides

Affine Variety

- Given an algebraic variety $\mathcal{A}$ over an algebraically closed field.
- The *ideal* I of $\mathcal{A}$ is the set of polynomials that vanish on $\mathcal{A}$.

$$I = \{f(x) \in \mathbb{F}[x] \mid f(x) = 0 \ \forall x \in \mathcal{A}\}.$$

- The *coordinate ring* $R[\mathcal{A}]$ of $\mathcal{A}$ is the quotient ring

$$R[\mathcal{A}] = \mathbb{F}[x]/I.$$

Hilbert Series

The *Hilbert Series* $\mathrm{HS}_{\mathcal{A}}(t)$ is the generating function that counts the number of holomorphic functions in the coordinate ring.:

$$\mathrm{HS}_{\mathcal{A}}(t) = \sum_{d=0}^{\infty} h_d t^d$$

where

- t is the counting variable.
- The coefficients h_d count linearly independent monomials of degree d in $R[\mathcal{A}]$. Equivalently, h_d count the gauge-invariant chiral operators of conformal dimensions d in field theory.